Model Answers.

ELIJAH MATHS
GCSE and A Level Maths

AS LEVEL
INTEGRATION

Question 1

$$f(x) = \frac{x^5 - 12x^{\frac{1}{2}}}{4x}$$

(a) Write f(x) in the form

$$ax^p + bx^q$$

where a, b, p and q are simplified constants.
(3)

(b) Hence find

$$\int f(x)\,dx$$

giving your answer in simplest form.
(3)

(a) $f(x) = \frac{x^5}{4x} - \frac{12x^{\frac{1}{2}}}{4x}$ ✓

$= \frac{1}{4}x^4 - 3x^{-\frac{1}{2}}$ ✓ ✓

3

(b) $\int \left(\frac{1}{4}x^4 - 3x^{-\frac{1}{2}}\right) dx$

$= \frac{\frac{1}{4}x^5}{5} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c$ ✓ ✓

3

$= \frac{1}{20}x^5 - 6x^{\frac{1}{2}} + c$ ✓

Question 2

Find

$$\int \frac{2\sqrt{x} - 3}{x^2} \, dx$$

giving your answer in simplest form.

(4)

$$\int \left(\frac{2x^{\frac{1}{2}}}{x^2} - \frac{3}{x^2} \right) dx$$

$$\Rightarrow \int \left(2x^{-\frac{3}{2}} - 3x^{-2} \right) dx \checkmark$$

$$\Rightarrow \frac{2x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{3x^{-1}}{-1} + c \checkmark$$

$$\Rightarrow -4x^{-\frac{1}{2}} + 3x^{-1} + c \checkmark$$

$$\Rightarrow \underline{\underline{-\frac{4}{\sqrt{x}} + \frac{3}{x} + c}} \checkmark$$

4

Question 3

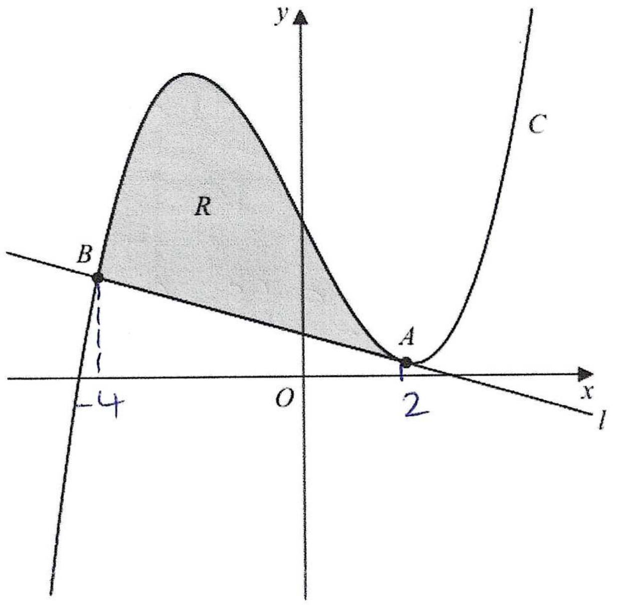

Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of the curve C with equation

$$y = x^3 - 14x + 23$$

The line l is the tangent to C at the point A, also shown in Figure 3.

Given that l has equation $y = -2x + 7$

(a) show, using calculus, that the x coordinate of A is 2

(3)

The line l cuts C again at the point B.

(b) Verify that the x coordinate of B is -4

(2)

The finite region, R, shown shaded in Figure 3, is bounded by C and l.

Using algebraic integration,

(c) show that the area of R is 108

(5)

(a) $\frac{dy}{dx} = 3x^2 - 14 = -2$ ✓

$\Rightarrow 3x^2 = 12$ ✓

$x^2 = 4$

$x = \cancel{4}\ 2$ ✓

3

(b) When $x = -4$

$y = -2x + 7 = -2(-4) + 7 = 15$

$y = x^3 - 14x + 23 = (-4)^3 - 14(-4) + 23$

$= 15$

As the y coordinates are the same the two graphs ~~the~~ intersect each other so x coordinate of B is -4

2

(c) Area = Curve − Line

Area ~~under curve~~ $= \int_{-4}^{2} x^3 - 14x + 23 - (-2x + 7)\,dx$

$\Rightarrow \int_{-4}^{2} (x^3 - 14x + 23 + 2x - 7)\,dx$

$\Rightarrow \int_{-4}^{2} (x^3 - 12x + 16)\,dx$

$\Rightarrow \left[\frac{1}{4}x^4 - 6x^2 + 16x\right]_{-4}^{2}$

$\Rightarrow \left[\frac{1}{4}(2)^4 - 6(2)^2 + 16(2)\right] -$

$\left[\frac{1}{4}(-4)^4 - 6(-4)^2 + 16(-4)\right]$

$= 108$

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Question 4

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

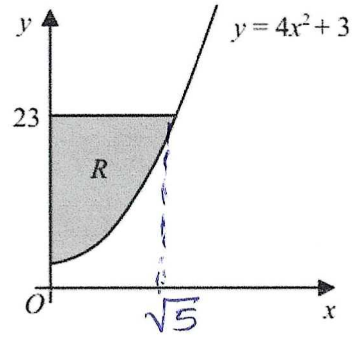

Figure 2

The finite region R, shown shaded in Figure 2, is bounded by the curve with equation $y = 4x^2 + 3$, the y-axis and the line with equation $y = 23$

Show that the exact area of R is $k\sqrt{5}$ where k is a rational constant to be found.

(5)

$4x^2 + 3 = 23$

$4x^2 = 20$

$x = \sqrt{5}$ ✓

Area of Rectangle $= 23\sqrt{5}$ ✓

Area under curve $= \int_0^{\sqrt{5}} (4x^2 + 3)\, dx$

$= \left[\frac{4x^3}{3} + 3x\right]_0^{\sqrt{5}}$ ✓

$= \frac{4(\sqrt{5})^3}{3} + 3\sqrt{5} = \frac{29\sqrt{5}}{3}$ ✓

Area of R $= 23\sqrt{5} - \frac{29\sqrt{5}}{3}$

$= \frac{40\sqrt{5}}{3}$ units^2 ✓

5

Question 5

. A curve has equation $y = f(x),\ x \geqslant 0$

Given that

- $f'(x) = 4x + a\sqrt{x} + b$, where a and b are constants
- the curve has a stationary point at $(4,3)$
- the curve meets the y-axis at -5

find $f(x)$, giving your answer in simplest form.

(6)

$f'(4) = 0;\ 4(4) + a\sqrt{4} + b = 0$

$16 + 2a + b = 0$

$2a + b = -16$ (1)

$f(x) = \int f'(x)\,dx = 2x^2 + \frac{2ax^{3/2}}{3} + bx + c$

at $(0,-5)$; $2(0)^2 + \frac{2a(0)^{\frac{3}{2}}}{3} + b(0) + c = -5$

$c = -5$

$f(x) = 2x^2 + \frac{2ax^{\frac{3}{2}}}{3} + bx - 5$

at $(4,3)$; $2(4)^2 + \frac{2a(\sqrt{4})^3}{3} + 4b - 5 = 3$

$\frac{16a}{3} + 4b = -24$ (11)

Sub (1) → (11)

$\frac{16a}{3} + 4(-2a - 16) = -24$

$\frac{16a}{3} - 8a - 64 = -24$

$-\frac{8}{3}a = 40$

$a = -15,\quad b = 14$

(6)

Question 6

Find

$$\int \left(8x^3 - \frac{3}{2\sqrt{x}} + 5 \right) \mathrm{d}x$$

giving your answer in simplest form.

(4)

$$\int \left(8x^3 - \frac{3}{2} x^{-\frac{1}{2}} + 5 \right) dx$$

$$= 2x^4 - 3x^{\frac{1}{2}} + 5x + c$$

Question 7

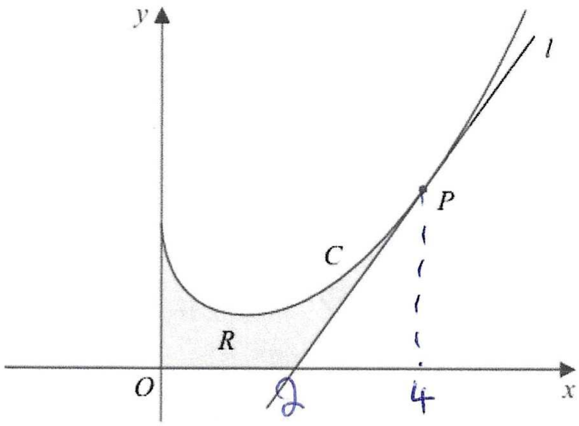

Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 2 shows a sketch of part of the curve C with equation

$$y = \frac{1}{3}x^2 - 2\sqrt{x} + 3 \qquad x \geqslant 0$$

The point P lies on C and has x coordinate 4

The line l is the tangent to C at P.

(a) Show that l has equation

$$13x - 6y - 26 = 0$$

(5)

The region R, shown shaded in Figure 2, is bounded by the y-axis, the curve C, the line l and the x-axis.

(b) Find the exact area of R.

(5)

(a) $y = \frac{1}{3}x^2 - 2x^{\frac{1}{2}} + 3$

$\frac{dy}{dx} = \frac{2}{3}x - x^{-\frac{1}{2}}$ ✓

@ $x = 4$: $\frac{dy}{dx} = \frac{2}{3}(4) - \frac{1}{\sqrt{4}}$

$= \frac{13}{6}$ (Gradient of tangent) ✓

$$y = \frac{1}{3}(4)^2 - 2\sqrt{4} + 3$$

$$= \frac{13}{3} \quad \therefore \quad P\left(4, \frac{13}{3}\right)$$

Equation of tangent:

$$y - \frac{13}{3} = \frac{13}{6}(x - 4)$$

(5)

$$6y - 26 = 13x - 52$$

$$\underline{13x - 6y - 26 = 0} \text{ as required.}$$

(b) Using equation of line; when $y = 0$:

$$13x = 26 \quad \therefore \quad x = 2$$

Area under curve $:= \int_0^4 \left(\frac{1}{3}x^2 - 2x^{\frac{1}{2}} + 3\right) dx$

$$= \left.\frac{1}{9}x^3 - \frac{4}{3}x^{3/2} + 3x\right]_0^4$$

$$= \left[\frac{1}{9}(4)^3 - \frac{4}{3}(4)^{\frac{3}{2}} + 3(4)\right] - 0$$

$$= \frac{76}{9}$$

Area of Triangle $= \frac{1}{2} \times 2 \times \frac{13}{3}$

$$= \frac{13}{3}$$

(5)

Area of R $= \frac{76}{9} - \frac{13}{3}$

$$= \underline{\frac{37}{9} \text{ units}^2}$$

Question 8

Find

$$\int \frac{3x^4 - 4}{2x^3}\,dx$$

writing your answer in simplest form.

(4)

$$\int \frac{3x^4}{2x^3} - \frac{4}{2x^3}\,dx = \int \frac{3}{2}x - 2x^{-3}\,dx$$

$$= \frac{3}{4}x^2 + x^{-2} + c$$

$$= \frac{3}{4}x^2 + \frac{1}{x^2} + c$$

4

Question 9

Find the value of the constant k, $0 < k < 9$, such that

$$\int_k^9 \frac{6}{\sqrt{x}}\,\mathrm{d}x = 20$$

(4)

$$= \int_k^9 6x^{-\frac{1}{2}}\,dx = 20$$

$$= \left[12x^{\frac{1}{2}}\right]_k^9 = 20 \quad ✓$$

$$\Rightarrow (12\sqrt{9}) - (12\sqrt{k}) = 20$$

$$\Rightarrow 36 - 12\sqrt{k} = 20 \quad ✓$$

$$\Rightarrow 12\sqrt{k} = 16$$

$$\sqrt{k} = \frac{16}{12}$$

$$\sqrt{k} = \frac{4}{3} \quad ✓$$

$$k = \frac{16}{9} \quad ✓$$

4

Question 10

. A curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = -3x^2 + 12x + 8$$

(a) Write $\mathrm{f}(x)$ in the form

$$a(x+b)^2 + c$$

where a, b and c are constants to be found.

(3)

The curve C has a maximum turning point at M.

(b) Find the coordinates of M.

(2)

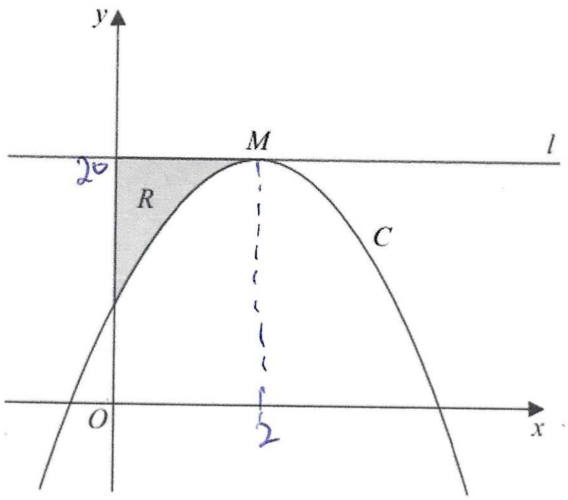

Figure 3

Figure 3 shows a sketch of the curve C.

The line l passes through M and is parallel to the x-axis.

The region R, shown shaded in Figure 3, is bounded by C, l and the y-axis.

(c) Using algebraic integration, find the area of R.

(5)

(a) $\mathrm{f}(x) = (-3x^2 + 12x) + 8$

$= -3[x^2 - 4x] + 8$ ✓

$= -3[(x-2)^2 - 4] + 8$

$= -3(x-2)^2 + 12 + 8$ ✓

$= -3(x-2)^2 + 20$ ✓

3

(b) $M(2, 20)$

(c) Equation of L: $y = 20$.

Area of rectangle; $= 2 \times 20$

$= 40$ units^2

Area under curve $= \int_0^2 -3x^2 + 12x + 8$

$= \left[-x^3 + 6x^2 + 8x\right]_0^2$

$= -(2)^3 + 6(2)^2 + 8(2) - 0$

$= 32$ unit^2

Area of R $= 40 - 32$

$= \underline{\underline{8 \text{ units}^2}}$

5

Question 11

Given that k is a positive constant and $\int_1^k \left(\frac{5}{2\sqrt{x}} + 3\right) \mathrm{d}x = 4$

(a) show that $3k + 5\sqrt{k} - 12 = 0$

(4)

(b) Hence, using algebra, find any values of k such that

$$\int_1^k \left(\frac{5}{2\sqrt{x}} + 3\right) \mathrm{d}x = 4$$

(4)

(a) $\int_1^k \left(\frac{5}{2}x^{-\frac{1}{2}} + 3\right) dx = 4$

$\Rightarrow \left[5x^{\frac{1}{2}} + 3x\right]_1^k = 4$ ✓

$\Rightarrow (5\sqrt{k} + 3k) - (5\sqrt{1} + 3(1)) = 4$ ✓

$\Rightarrow 5\sqrt{k} + 3k - 8 = 4$ ✓

$\Rightarrow \underline{\underline{3k + 5\sqrt{k} - 12 = 0}}$ ✓ 4

(b) let $x = \sqrt{k}$

$3x^2 + 5x - 12 = 0$ ✓

$\frac{(3x+9)(3x-4)}{3} = 0$

$(x+3)(3x-4) = 0$ ✓

$x = -3, \quad x = \frac{4}{3}$ ✓

$\sqrt{k} = -3$ or $\sqrt{k} = \frac{4}{3}$ 4

$k = \frac{16}{9}$ ✓

Question 12

$$g(x) = 2x^3 + x^2 - 41x - 70$$

(a) Use the factor theorem to show that g(x) is divisible by (x – 5).

(2)

(b) Hence, showing all your working, write g(x) as a product of three linear factors.

(4)

The finite region R is bounded by the curve with equation $y = g(x)$ and the x-axis, and lies below the x-axis.

(c) Find, using algebraic integration, the exact value of the area of R.

(4)

(a) $x - 5 = 0 \Rightarrow x = 5$

$g(5) = 2(5)^3 + 5^2 - 41(5) - 70 = 0$

Since $g(5) = 0 \Rightarrow x - 5$ is a factor of $g(x)$

2

(b) $2x^3 + x^2 - 41x - 70 = (x-5)(ax^2 + bx + c)$

$= ax^3 + bx^2 + cx - 5ax^2 - 5bx - 5c$

$= ax^3 + (b - 5a)x^2 + (c - 5b)x - 5c$

Compare co-efficients

$a = 2$ $\quad$ $b - 5a = 1$ $\quad$ $-5c = -70$

$b - 10 = 1$

$b = 11$ $\quad$ $c = 14$

$\Rightarrow 2x^3 + x^2 - 41x - 70 = (x-5)(2x^2 + 11x + 14)$

$= \underline{(x-5)(2x+7)(x+2)}$

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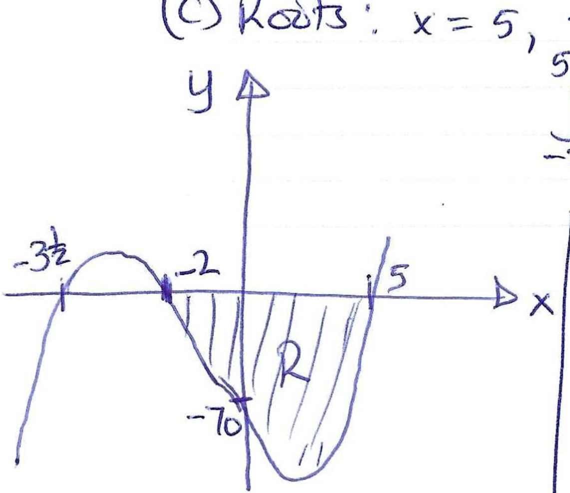
(c) Roots: $x = 5, -3\frac{1}{2}, -2$, y-intercept $= -70$



$\int_{-2}^{5} (2x^3 + x^2 - 41x - 70)\,dx$

$= \left[\frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{41}{2}x^2 - 70x\right]_{-2}^{5}$

$= \left[\frac{1}{2}(5)^4 + \frac{1}{3}(5)^3 - \frac{41}{2}(5)^2 - 70(5)\right]$

$- \left[\frac{1}{2}(-2)^4 + \frac{1}{3}(-2)^3 - \frac{41}{2}(-2)^2 - 70(-2)\right]$

$= -\frac{1715}{3}$

$\Rightarrow$ Area $= \frac{1715}{3}$ units^2

4

Question 13

(a) Given that k is a constant, find

$$\int\left(\frac{4}{x^3}+kx\right)\mathrm{d}x$$

simplifying your answer.

(3)

(b) Hence find the value of k such that

$$\int_{0.5}^{2}\left(\frac{4}{x^3}+kx\right)\mathrm{d}x = 8$$

(3)

(a) $\int(4x^{-3}+kx)\,dx = -2x^{-2}+\frac{kx^2}{2}$

$= -\frac{2}{x^2}+\frac{kx^2}{2}$

3

(b) $\left[\frac{kx^2}{2}-\frac{2}{x^2}\right]_{0.5}^{2} = 8$

$\Rightarrow (2k-0.5)-\left(\frac{1}{8}k-8\right) = 8$

$\Rightarrow \frac{15}{8}k+\frac{15}{2} = 8$

$\Rightarrow 15k+60 = 64$

$15k = 4$

$k = \frac{4}{15}$

3

Question 14

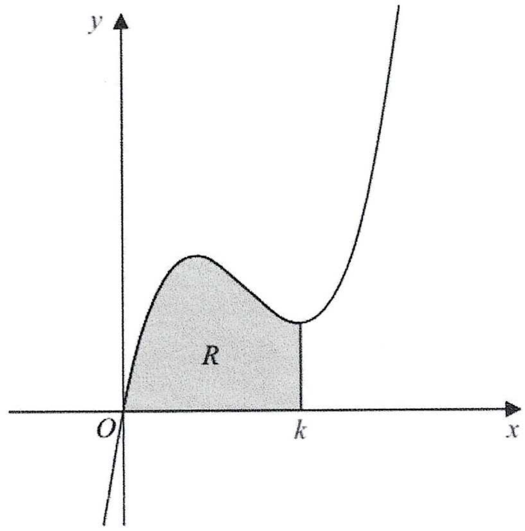


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2x^3 - 17x^2 + 40x$$

The curve has a minimum turning point at $x = k$.

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and the line with equation $x = k$.

Show that the area of R is $\frac{256}{3}$

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(7)

$\frac{dy}{dx} = 6x^2 - 34x + 40 = 0$

$3x^2 - 17x + 20 = 0$

$(3x - 12)(3x - 5) = 0$

$(x - 4)(3x - 5) = 0$

$x = 4, \quad x = \frac{5}{3}$ $\quad \boxed{x = 4}$

$\text{Area} = \int_0^4 (2x^3 - 17x^2 + 40x)\,dx$

$= \left[\frac{1}{2}x^4 - \frac{17x^3}{3} + 20x^2\right]_0^4$

$= \left[\frac{1}{2}(4)^4 - \frac{17}{3}(4)^3 + 20(4)^2\right] - 0$ $\quad \boxed{7}$

$= \frac{256}{3}$ units^2