Model Answers

ELIJAH MATHS — GCSE and A Level Maths

as level

DIFFERENTIATION

Question 1

$$f(x) = ax^3 + bx^2 + 18x + 9$$

where a and b are constants.

Given that $(x + 3)$ is a factor of $f(x)$

(a) show that

$$-3a + b = 5$$

(2)

Given also that $f'(2) = 14$

(b) find the value of a and the value of b.

(5)

(a) If $(x+3)$ is a factor of $f(x)$, then $f(-3) = 0$

$(-3)^3 a + (-3)^2 b + 18(-3) + 9 = 0$ ✓

$-27a + 9b - 45 = 0$

$-3a + b = 5$ as required ✓ (2)

(b) $f'(x) = 3ax^2 + 2bx + 18$ ✓

$f'(2) = 3a(4) + 2b(2) + 18 = 14$ ✓

$\Rightarrow 12a + 4b = -4$

$-3a + b = 5$) ×4

$12a + 4b = -4$ ✓ (5)

$-12a + 4b = 20$

$8b = 16$

$b = 2$ ✓

$a = -1$ ✓

Question 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

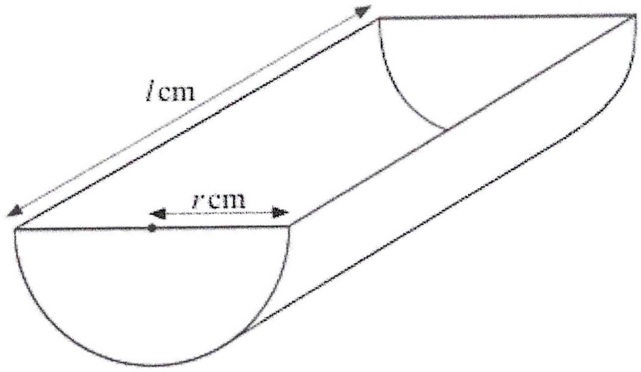

Figure 4

Figure 4 shows a design for a feeding trough.

The trough is modelled as a hollow, semicircular cylinder of radius r cm and length l cm.

The trough will be made from sheet metal of negligible thickness.

Given that the capacity of the trough will be $90\,000\pi\,\text{cm}^3$

(a) show that the total area, $A\,\text{cm}^2$, of sheet metal required to make the trough is given by

$$A = \frac{180\,000\pi}{r} + \pi r^2$$

(4)

(b) Use calculus to find the radius of the trough for which A is a minimum.

(4)

(c) Show that the radius found in part (b) gives the minimum value of A.

(2)

Given that the sheet metal costs £30 per square metre

(d) calculate the minimum cost of sheet metal required to make one trough.

(2)

(e) State one assumption you have made in calculating your answer to part (d).

(1)

(a) $V = \pi r^2 h$.

$V = \pi r^2 h = 180\,000$ ✓

$h = \dfrac{180\,000}{\pi r^2}$ ✓

$A = \pi r^2 + \pi r l$

$= \pi r^2 + \pi r \left(\frac{180\,000}{r^2}\right)$ ✓

$A = \frac{180\,000\pi}{r} + \pi r^2$ ✓ as required. (4)

(b) $A = 180000\pi r^{-1} + \pi r^2$ ✓

$\frac{dA}{dr} = -\frac{180\,000\pi}{r^2} + 2\pi r$ ✓ $= 0$ (at min)

$2\pi r^3 - 180000\pi = 0$ ✓

$r^3 = \frac{180000\pi}{2\pi}$

$r = 44.8\text{cm}$ ✓ (4)

(c) $\frac{d^2A}{dr^2} = \frac{360\,000\pi}{r^3} + 2\pi$ ✓

at $r = 44.8$: $\frac{d^2A}{dr^2} = \frac{360000}{(44.8)^3} + 2\pi$ (2)

$= 6\pi > 0$ ✓ hence min.

(d) at $r = 44.8$: $A = \frac{180\,000\pi}{44.8} + \pi \times 44.8^2$ ✓

$= 18928\text{ cm}^2 = 1.8928\text{m}^2$

Cost $= 30 \times 1.8928 = £56.78$ ✓ (2)

(e) No metal is wasted eg cutting semi circular pieces from rectangular sheets. ✓ (1)

Question 3

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

The growth of a particular tree is monitored over a period of time.

The height, h metres, of this tree, t years after it was planted, is modelled by the equation

$$h = 31 - Ae^{-kt}$$

where A and k are positive constants.

Given that

- exactly 10 years after it was planted, the height of the tree was 6 m
- exactly 20 years after it was planted, the height of the tree was 11 m

(a) find a complete equation for h in terms of t, giving the value of each of A and k to 3 significant figures.

(4)

Use the equation of the model to answer parts (b), (c) and (d).

According to the model, there is a limit to the height to which this tree can grow.

(b) Deduce this limit.

(1)

(c) (i) Find the initial height of the tree.

(ii) Hence explain whether this is a suitable model for the early growth of the tree.

(2)

(d) (i) Find $\frac{dh}{dt}$, giving your answer in simplest form.

(2)

(ii) Hence find the value of t for which the height of the tree is increasing at a rate of 30 cm per year.

(3)

(a) $h=6$, $t=10$: $6 = 31 - Ae^{-10k}$
$Ae^{-10k} = 25$ ① ✓

$h=11$, $t=20$: $11 = 31 - Ae^{-20k}$
$Ae^{-20k} = 20$ ② ✓

① ÷ ②: $\frac{Ae^{-10k}}{Ae^{-20k}} = \frac{25}{20}$ ∴ $e^{10k} = 5/4$ ✓

(4)

$10k = \ln\left(\frac{5}{4}\right) \Rightarrow k = \frac{1}{10}\ln(5/4) = 0.0223$

Sub into ①: $Ae^{0.223} = 25 \Rightarrow A = 31.3$

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∴ $h = 31 - 31.3e^{-0.0223t}$ ✓

(b) 31m. ✓ (1)

(c)(i) $31 - 31.3 = -0.3\text{m}$ ✓

(ii) not suitable as prediction is a negative height. ✓ (2)

d(i) $\frac{dh}{dt} = 31.3 \times 0.0223 e^{-0.0223t}$ ✓

$= 0.698 e^{-0.0223t}$ ✓ (2)

(ii) $0.698 e^{-0.0223t} = 0.3$ ✓

$e^{-0.0223t} = 0.42980$ ✓

$-0.0223t = \ln(0.42980)$

$t = 37.9$ ✓ (3)

Question 4

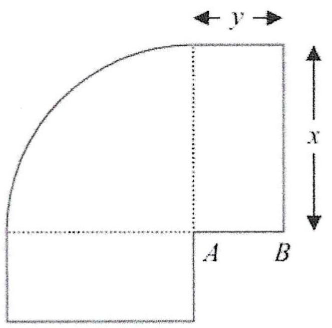

Figure 5

Figure 5 shows the plan view of the design for a swimming pool.

The pool is modelled as a quarter of a circle joined to two equal sized rectangles as shown.

Given that

- the quarter circle has radius x metres
- the rectangles each have length x metres and width y metres
- the total surface area of the swimming pool is $100\,\text{m}^2$

(a) show that, according to the model, the perimeter P metres of the swimming pool is given by

$$P = 2x + \frac{200}{x}$$

(5)

(b) Use calculus to find the value of x for which P has a stationary value.

(4)

(c) Prove, by further calculus, that this value of x gives a minimum value for P

(2)

Access to the pool is by side AB shown in Figure 5.

Given that AB must be at least one metre,

(d) determine, according to the model, whether the swimming pool with the minimum perimeter would be suitable.

(2)

(a) surface area $= 2xy + \frac{\pi x^2}{4} = 100$ ✓

$8xy + \pi x^2 = 400$

$y = \frac{400 - \pi x^2}{8x}$ ✓

Perimeter $= 2x + 4y + \frac{2\pi x}{4}$

$\Rightarrow 2x + 4\left(\frac{400 - \pi x^2}{8x}\right) + \frac{2\pi x}{4}$ ✓

$= 2x + \frac{400}{2x} - \frac{\pi x}{2} + \frac{\pi x}{2}$ ✓ (5)

$= 2x + \frac{200}{x}$ shown ✓

(b) $P = 2x + \frac{200}{x} \Rightarrow 2x + 200x^{-1}$ ✓

$\frac{dP}{dx} = 2 - \frac{200}{x^2} = 0$ ✓

$2x^2 - 200 = 0$ ✓ (4)

$x^2 = 100$

$x = 10$ ✓

(c) $\frac{dP}{dx} = 2 - 200x^{-2} \Rightarrow \frac{d^2P}{dx^2} = \frac{400}{x^3}$ ✓ (2)

When $x = 10$: $\frac{d^2P}{dx^2} = \frac{400}{1000} = 4 > 0$ ✓ min

(d) when $x = 10$: $y = \frac{400 - \pi \times 10^2}{8 \times 10}$ ✓

$= 1.073\text{m} > 1\text{m}$

yes suitable. ✓ (2)

Question 5

A curve has equation

$$y = \frac{2}{3}x^3 - \frac{7}{2}x^2 - 4x + 5$$

(a) Find $\frac{dy}{dx}$ writing your answer in simplest form.

(2)

(b) Hence find the range of values of x for which y is decreasing.

(4)

(a) $\frac{dy}{dx} = 2x^2 - 7x - 4$ ✓✓ (2)

(b) $2x^2 - 7x - 4 < 0$ ✓

$(2x - 8)(2x + 1) < 0$

$(x - 4)(2x + 1) < 0$ ✓

$x = 4, \quad x = -\frac{1}{2}$ ✓ (4)

y is decreasing for $-\frac{1}{2} < x < 4$ ✓

Question 6

. A company makes drinks containers out of metal.

The containers are modelled as closed cylinders with base radius r cm and height h cm and the capacity of each container is $355\,\text{cm}^3$

The metal used

- for the circular base and the curved side costs 0.04 pence/cm^2
- for the circular top costs 0.09 pence/cm^2

Both metals used are of negligible thickness.

(a) Show that the total cost, C pence, of the metal for one container is given by

$$C = 0.13\pi r^2 + \frac{28.4}{r}$$

(4)

(b) Use calculus to find the value of r for which C is a minimum, giving your answer to 3 significant figures.

(4)

(c) Using $\frac{d^2C}{dr^2}$ prove that the cost is minimised for the value of r found in part (b).

(2)

(d) Hence find the minimum value of C, giving your answer to the nearest integer.

(2)

(a) $V = \pi r^2 h \Rightarrow \pi r^2 h = 355$

$\therefore h = \frac{355}{\pi r^2}$ ✓

Cost of base $= 0.04\pi r^2$

Cost of side $= 0.04 \times 2\pi r h$

$= 0.04 \times 2\pi r \left(\frac{355}{\pi r^2}\right)$

$= \frac{28.4}{r}$ ✓

Cost of top $= 0.09\pi r^2$

✓

Total cost $= 0.04\pi r^2 + \frac{28.4}{r} + 0.09\pi r^2$

$C = 0.13\pi r^2 + \frac{28.4}{r}$ ✓ (4)

(b) $C = 0.13\pi r^2 + 28.4r^{-1}$

$\frac{dC}{dr} = 0.26\pi r - \frac{28.4}{r^2} = 0$ ✓

$0.26\pi r = \frac{28.4}{r^2}$ ✓

$r^3 = \frac{28.4}{0.26\pi}$ ✓ (4)

$r = 3.26$ cm. ✓

(c) $\frac{dC}{dr} = 0.26\pi r - 28.4r^{-2}$

$\frac{d^2C}{dr^2} = 0.26\pi + \frac{56.8}{r^3}$

For $r = 3.26$: $\frac{d^2C}{dr^2} = 0.26\pi + \frac{56.8}{(3.26)^3}$ ✓

$= 2.46 > 0$ ✓ (2)

hence minimised

(d) $r = 3.2$: $C = 0.13\pi(3.26)^2 + \frac{28.4}{3.26}$ ✓

$= 13.05$ (2)

Minimum cost = 13p. ✓

Question 7

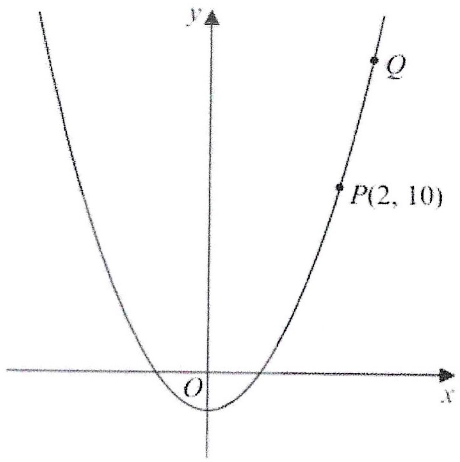

Figure 1

Figure 1 shows part of the curve with equation $y = 3x^2 - 2$

The point $P(2, 10)$ lies on the curve.

(a) Find the gradient of the tangent to the curve at P.

(2)

The point Q with x coordinate $2 + h$ also lies on the curve.

(b) Find the gradient of the line PQ, giving your answer in terms of h in simplest form.

(3)

(c) Explain briefly the relationship between part (b) and the answer to part (a).

(1)

(a) $\frac{dy}{dx} = 6x$ ✓

at $P(2,10)$: $\frac{dy}{dx} = 6 \times 2$

$= 12$ ✓ (2)

(b) $m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{3(2+h)^2 - 2 - 10}{(2+h) - 2}$ ✓

$= \frac{3(4 + 4h + h^2) - 12}{h}$ ✓

$= \frac{12 + 12h + 3h^2 - 12}{h}$

$\frac{12h + 3h^2}{h} = 12 + 3h$ ✓ (3)

(c) As $h \to 0$, $12 + 3h \to 12$ ✓

$\therefore$ Gradient of chord tends to gradient of tangent to curve. (1)

Question 8

The curve C has equation $y = f(x)$ where

$$f(x) = ax^3 + 15x^2 - 39x + b$$

and a and b are constants.

Given

- the point (2, 10) lies on C
- the gradient of the curve at (2, 10) is –3

(a) (i) show that the value of a is –2

(ii) find the value of b.

(4)

(b) Hence show that C has no stationary points.

(3)

(c) Write f(x) in the form $(x - 4)Q(x)$ where $Q(x)$ is a quadratic expression to be found.

(2)

(d) Hence deduce the coordinates of the points of intersection of the curve with equation

$$y = f(0.2x)$$

and the coordinate axes.

(2)

(a) $f(x) = ax^3 + 15x^2 - 39x + b$

$f'(x) = 3ax^2 + 30x - 39$

$f(2) = 10;\ a(2)^3 + 15(2)^2 - 39(2) + b = 10$

$8a + 60 - 78 + b = 10$ ✓

$\therefore\ 8a + b = 28$ (i)

$f'(2) = -3;\ 3a(2)^2 + 30(2) - 39 = -3$ ✓

$12a + 60 - 39 = -3$

$12a = -24$

(i) $a = -2$ (ii) ✓

(ii) Sub (ii) → (i); $-16 + b = 28$

$b = 44$ ✓ (4)

(b) $f(x) = -2x^3 + 15x^2 - 39x + 44$

$f'(x) = -6x^2 + 30x - 39$ ✓

$b^2 - 4ac < 0 \Rightarrow$ No stationary points (3)

$\Rightarrow 30^2 - 4(-6)(-39) = -36 < 0$ ✓ $\therefore$ ✓ no stationary points.

(c) $-2x^3 + 15x^2 - 39x + 44 = (x-4)(ax^2 + bx + c)$

$= ax^3 + bx^2 + cx - 4ax^2 - 4bx - 4c$

$= ax^3 + (b-4a)x^2 + (c-4b)x - 4c$ ✓

Compare Co-efficients.

$a = -2$, $b - 4a = 15$, $-4c = 44$

$b + 8 = 15$

$b = 7$, $c = -11$

$\therefore -2x^3 + 15x^2 - 39x + 44 = (x-4)(-2x^2 + 7x - 11)$ ✓ (2)

(d) Where it crosses y-axis, $x = 0$

$f(0) = 44 \Rightarrow (0, 44)$

Where it crosses x-axis, $y = 0$

$f(x) = 0$; $(x-4)(-2x^2 + 7x - 11) = 0$

$x = 4$ (only real solution

$\therefore (4, 0)$

$f(x) \rightarrow f(\frac{1}{5}x)$ is a stretch factor 5 in x-direction.

$(0, 44) \rightarrow [(0 \times 5), 44] = (0, 44)$ ✓

$(4, 0) \rightarrow (4 \times 5, 0) = (20, 0)$ ✓ (2)

Question 9

A curve has equation

$$y = 3x^2 + \frac{24}{x} + 2 \qquad x > 0$$

(a) Find, in simplest form, $\frac{dy}{dx}$

$y = 3x^2 + 24x^{-1}$ ✓

(3)

(b) Hence find the exact range of values of x for which the curve is increasing.

(2)

(a) $\frac{dy}{dx} = 6x - \frac{24}{x^2}$ ✓ ✓ (3)

(b) For increasing function, $\frac{dy}{dx} > 0$

$6x - \frac{24}{x^2} > 0$

$6x^3 - 24 > 0$

$6x^3 > 24$ ✓

$x^3 > 4$

$x > \sqrt[3]{4}$ ✓ (2)

Question 10

Prove, from first principles, that the derivative of x^3 is $3x^2$

$$f(x+h) = (x+h)^3$$
$$= x^3 + 3x^2h + 3xh^2 + h^3 \checkmark$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \checkmark$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 \checkmark$$

(4)

$$= 3x^2 \checkmark$$